Julia sets in higher dimensions

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Overview

- Complex dynamics.
- Quasiregular maps on $\mathbb{R}^d$.
- Iteration of quasiregular maps.

Complex dynamics

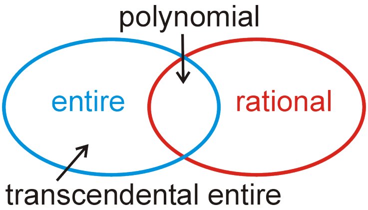
- The study of iteration of analytic maps on $\mathbb{C}$ (or $\mathbb{C} \cup \{\infty\}$).

- Interested in the long-term behaviour of the sequence

$$z, \;\; f(z), \;\; f^2(z), \;\; f^3(z), \;\; \ldots$$

  for different starting values z. Notation: e.g., $f^2(z) = f(f(z))$.

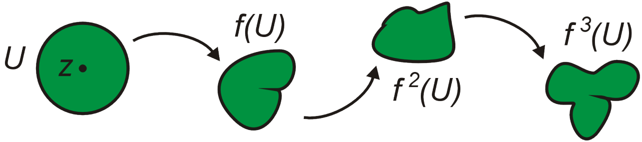
- Can consider different types of functions:



Complex dynamics

Given $f: \mathbb{C} \to \mathbb{C}$, we partition the plane into the *Fatou set* $\mathcal{F}_f$ and the *Julia set* $\mathcal{J}_f$.

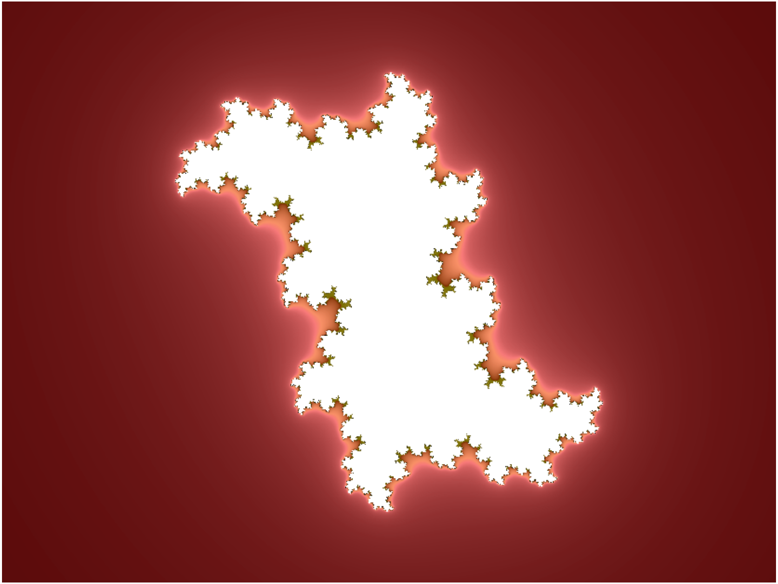
Fatou set Set of starting values z near which iterates f^n are 'stable'.
If U is a small disc in $\mathcal{F}_f$, then $f^n(U)$ always stays small.



Formally: $\mathcal{F}_f = \{z : \{f^n\} \text{ is normal on a nhd of } z\}$.

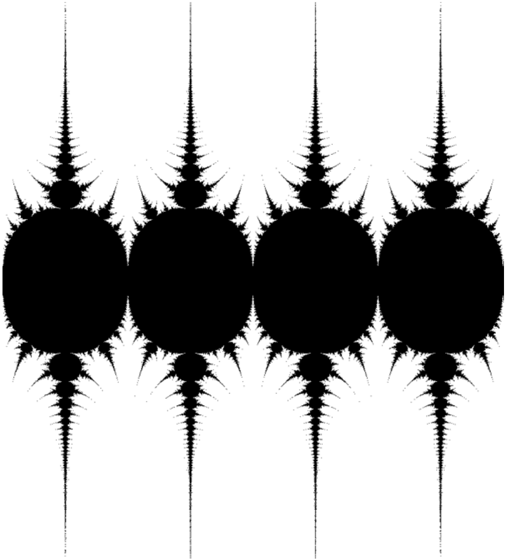
Julia set $\mathcal{J}_f = \mathbb{C} \setminus \mathcal{F}_f$.
The iterates f^n behave chaotically on $\mathcal{J}_f$.
For $z \in \mathcal{J}_f$, can find w arbitrarily close to z such that sequences $(f^n(z))$ and $(f^n(w))$ are very different.

$f(z) = z^2 + 0.18 + 0.55i$. The Julia set $\mathcal{J}_f$ is boundary of white region.



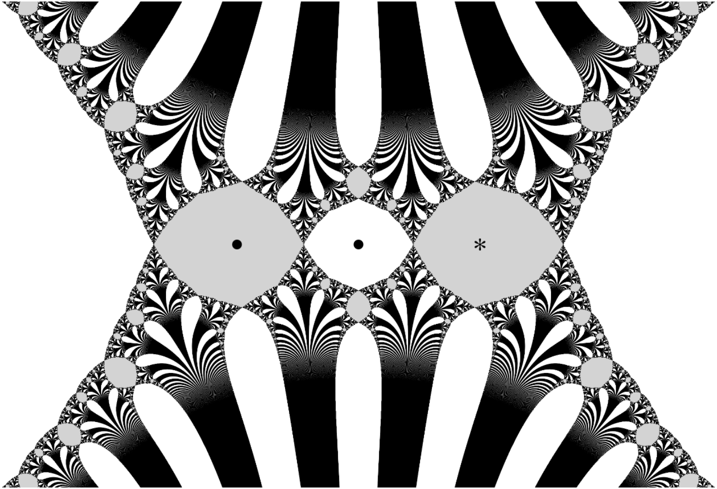
Picture credit: `usefuljs.net/fractals`

$f(z) = \sin z$. The Julia set $\mathcal{J}_f$ is boundary of black region.



Picture credit: `http://paulbourke.net/fractals/sinjulia/`

- $\mathcal{J}_{e^z} = \mathbb{C}$.
- $f(z) = -z^2 e^{1-z^2}$. Julia set $\mathcal{J}_f$ is in black.



Superattracting fixed points at 0 and -1 shown as dots.

Picture credit: L. Rempe-Gillen

Properties of Julia sets

Let f be a complex polynomial of degree ≥ 2. Then

- $\mathcal{J}_f$ is closed and non-empty.
- $\mathcal{J}_f$ is *completely invariant under* f: that is,

$$f(\mathcal{J}_f) = \mathcal{J}_f = f^{-1}(\mathcal{J}_f).$$

- *Blowing-up property:* if U is open and meets $\mathcal{J}_f$, then

$$\bigcup_{n\geq 1} f^n(U) \supset \mathbb{C} \setminus \{\text{one point}\}.$$

- If $z \in \mathcal{J}_f$, then $\mathcal{J}_f = \overline{\bigcup_{n\geq 0} f^{-n}(z)}$.
- $\mathcal{J}_f$ has a dense subset of periodic points.

Similar results hold for rational and transcendental entire functions.

Quasiregular mappings

- Quasiregular maps $\mathbb{R}^d \to \mathbb{R}^d$ generalise analytic functions on $\mathbb{C}$.
- Analytic functions map small circles to small circles.
- Informally, a continuous map $f\colon \mathbb{R}^d \to \mathbb{R}^d$ is called *quasiregular* (qr) if it maps infinitesimal spheres to infinitesimal ellipsoids of bounded eccentricity (i.e. the ratio major axis/minor axis is bdd).
- For $K \geq 1$, we say that f is K-quasiregular if the amount of local stretching is $\leq K$ everywhere.

Simple examples

- $(x, y) \mapsto (Kx, y)$ is K-qr.
- With $N \in \mathbb{N}$ and cyl. co-ords on $\mathbb{R}^3$, the map $(r, \theta, x_3) \mapsto (r, N\theta, x_3)$.
- Analytic functions on $\mathbb{C}$ are 1-qr.

Qr maps of $\mathbb{R}^d$ classified as *polynomial type* or *transcendental type*.

Quasiregular mappings

Good news

- Some machinery from complex analysis carries over to qr maps.
- Qr maps are open, discrete and differentiable a.e.
- Composition of two quasiregular maps is itself quasiregular.
  Thus: f is qr $\implies$ f^n is qr.

Bad news

- Sum of two quasiregular maps need not be quasiregular.
- The amount of stretching grows on iteration:
  f is K-qr $\implies$ only that f^n is K^n-qr.

Example: the Zorich mapping

The Zorich map $Z\colon \mathbb{R}^3 \to \mathbb{R}^3 \setminus \{\mathbf{0}\}$ is a quasiregular analogue of the exponential function. It can be defined as follows:

1. Choose a bi-Lipschitz map

$$h\colon [-1,1]^2 \to \{(x_1,x_2,x_3) : x_1^2+x_2^2+x_3^2=1,\ x_3 \geq 0\}.$$

2. Define $Z\colon [-1,1]^2 \times \mathbb{R} \to \{(x_1,x_2,x_3) : x_3 \geq 0\}$ by

$$Z(x_1,x_2,x_3) = e^{x_3} h(x_1,x_2).$$

3. Extend Z to all of $\mathbb{R}^3$ by repeatedly reflecting in planes.

The Zorich map is quasiregular on $\mathbb{R}^3$, is periodic in the x_1 and x_2 directions, and grows/decays exponentially in x_3 direction.

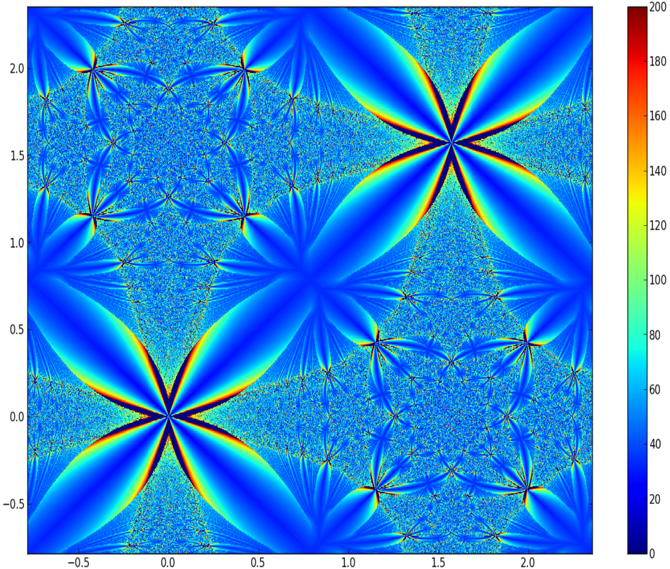
More examples

- For $z \in \mathbb{C}$, $p \in \mathbb{N}$, recall that $z^p = \exp(p \log z)$.
  Similarly, can define a quasiregular "power map" by
  $$x \mapsto Z(pZ^{-1}(x)).$$

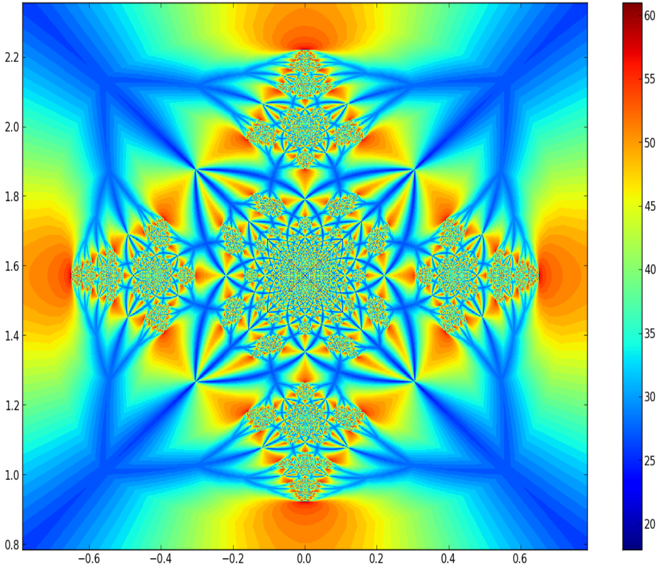
- A quasiregular "sine" function $S\colon \mathbb{R}^3 \to \mathbb{R}^3$ can be constructed by mapping a *half*-infinite beam onto $\{x_3 \geq 0\}$, then extending by reflections.

- Can construct a "tangent" function $T\colon \mathbb{R}^3 \to \mathbb{R}^3 \cup \{\infty\}$ analogous to $\tan z$.
  Numerical plots of iteration of T follow...

Iteration of T. Blue points $\to$ **0** fast, red points $\to$ **0** slowly.



Iteration of $0.7T$ near a pole. Thanks to Dan Goodman for code.



Are there Julia sets for qr maps?

- The classical "non-normality of $\{f^n\}$" definition of $\mathcal{J}_f$ isn't helpful for general quasiregular maps.

- Instead, use blowing-up property to *define* a Julia set.
  Sun and Yang suggested, for quasiregular maps on $\mathbb{C}$ $(= \mathbb{R}^2)$, defining

$$\mathcal{J}_f = \{z \in \mathbb{C} : \text{for any nhd } U \text{ of } z, \bigcup_{n \geq 1} f^n(U) \supset \mathbb{C} \setminus \{\text{one point}\}\}$$

Theorem (Sun and Yang, c.2000)

If $f : \mathbb{C} \to \mathbb{C}$ is K-qr with $K < \deg(f) < \infty$, then $\mathcal{J}_f \neq \emptyset$.

Can this be generalised further?

Ideally, we might consider

$$\mathcal{J}_{finite} = \{x \in \mathbb{R}^d : \text{for any nhd } U \text{ of } x, \bigcup_{n\geq 1} f^n(U) \supset \mathbb{R}^d \setminus \{\text{finite set}\}\}$$

but we don't yet know this is always non-empty. Instead, we allow a 'small' infinite set to be missed out...

Theorem (Bergweiler (deg $< \infty$), Bergweiler and N. (deg $= \infty$))

Let $f : \mathbb{R}^d \to \mathbb{R}^d$ *be* K*-quasiregular, with* $K < \deg(f) \leq \infty$. *Define*

$$\mathcal{J}_{cap} = \{x \in \mathbb{R}^d : \textit{for any nhd } U \textit{ of } x, \bigcup_{n\geq 1} f^n(U) \supset \mathbb{R}^d \setminus \{\textit{small set}\}\}.$$

Then $\mathcal{J}_{cap} \neq \emptyset$.

- Here $\{\textit{small set}\}$ means a set of conformal capacity zero.
- All definitions of $\mathcal{J}$ agree for analytic functions on $\mathbb{C}$.
- Conjecture that $\mathcal{J}_{cap} = \mathcal{J}_{finite}$.

Iteration of quasiregular “sine” function

Theorem (Fletcher and N.)

For a quasiregular analogue of the sine function, $S\colon \mathbb{R}^d \to \mathbb{R}^d$,

- *for every non-empty open set U, we have*

$$\bigcup_{n\geq 1} S^n(U) = \mathbb{R}^d;$$

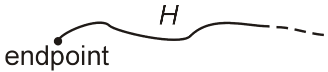
- *thus all points have the blowing-up property, so $\mathcal{J}_S = \mathbb{R}^d$.*
- *Periodic points of S are dense in $\mathbb{R}^d$.*

An application: Karpińska's paradox

Bergweiler and Eremenko introduced and iterated the qr "sine" map S to create a counter-intuitive decomposition of $\mathbb{R}^d$.

Definition

A subset $H \subset \mathbb{R}^d$ is called a *hair* if there is a continuous bijection $\gamma \colon [0, \infty) \to H$ such that $\gamma(t) \to \infty$ as $t \to \infty$. We call $\gamma(0)$ the *endpoint* of the hair.



Bergweiler and Eremenko expressed $\mathbb{R}^d$ as an (uncountable) union of hairs such that

- any two hairs intersect only at a common endpoint (if at all); and
- the union of the hairs without their endpoints has Hausdorff dimension 1. (It follows that set of endpoints has Hausdorff dim d.)

Escape to infinity

Definition

For a function $f\colon \mathbb{R}^d \to \mathbb{R}^d$, the *escaping set* is

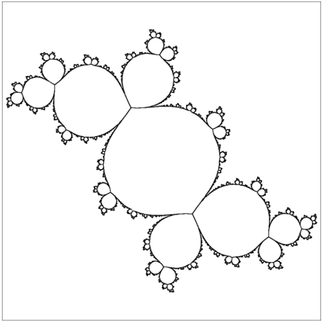
$$I(f) = \{x \in \mathbb{R}^d : f^n(x) \to \infty\}.$$

<u>For a complex polynomial f:</u>

- All large z belong to $I(f)$;
- all escaping points $\to \infty$ at same rate;
- $\mathcal{J}_f = \partial I(f)$.

<u>For polynomial type qr maps:</u>

- same as above, except only get $\mathcal{J}_f \subset \partial I(f)$.



Escape to infinity

For transcendental entire functions on $\mathbb{C}$:

- Eremenko (1989) proved that $I(f) \neq \emptyset$ and $\mathcal{J}_f = \partial I(f)$.
- Big open question: is every component of $I(f)$ unbounded?
- Escaping points can $\to \infty$ at different rates.
- There is a *fast escaping set* $A(f) \subset I(f)$. Again, $\mathcal{J}_f = \partial A(f)$.

For transcendental type qr maps on $\mathbb{R}^d$:

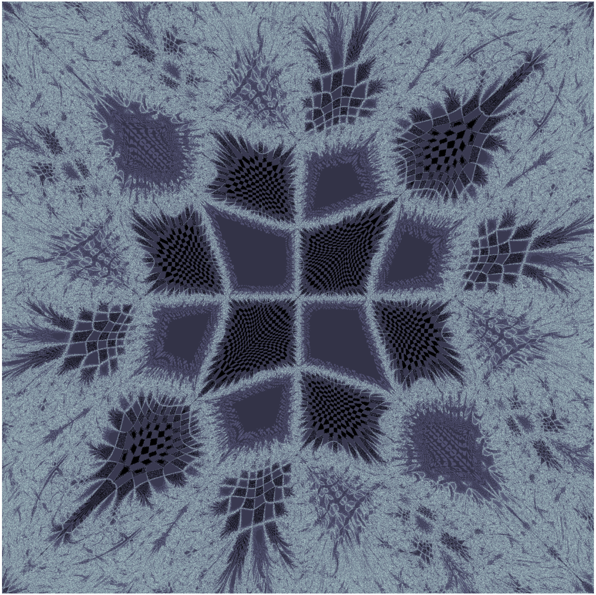
- Escaping set $I(f) \neq \emptyset$, but only have inclusion $\mathcal{J}_f \subset \partial I(f)$.
- There is an example for which $\mathcal{J}_f \neq \partial I(f)$ and another for which $\overline{I(f)}$ has a bounded component.

Theorem (Bergweiler, Fletcher and N.)

If f is trans type qr and does not grow too slowly, then $\mathcal{J}_f = \partial A(f)$.

Conjecture: This holds without the growth condition.

Escaping points of the Zorich mapping:



Picture credit: A. Fletcher and D. Goodman